How to correctly prune tropical trees

Jean-Vincent Loddo
Luca Saiu

http://www-lipn.univ-paris13.fr/~loddo
http://www-lipn.univ-paris13.fr/~saiu

LIPN, Université Paris 13

Paris, July 5, 2010

Motivation

Many problems can be solved with a non-deterministic search on a space where some metric is optimized.

Motivation

Many problems can be solved with a non-deterministic search on a space where some metric is optimized.

- Tropical games are a generalization of min-max games

Motivation

Many problems can be solved with a non-deterministic search on a space where some metric is optimized.

- Tropical games are a generalization of min-max games
- Tropical games support tropical α-pruning
    - tropical games whose dual is also tropical can be computed with α-β (not proved formally, but intuitive)

Motivation

Many problems can be solved with a non-deterministic search on a space where some metric is optimized.

- Tropical games are a generalization of min-max games
- Tropical games support tropical α-pruning
  - tropical games whose dual is also tropical can be computed with α-β (not proved formally, but intuitive)
- *Approximated parsing* is a tropical game
  - α-pruning seems to be effective
  - ...we suspect many more problems to be tropical games

Combinatorial games (1): min-max

Let a game arena $S = (\mathbb{P}, \lambda, succ)$ and a payoff function $p : \mathbb{P}_T \to \mathbb{Z} \cup \{+\infty, -\infty\}$ be given.

The *game value* of a position $\pi \in \mathbb{P}$ is traditionally defined as:

$$v_p(\pi) = \begin{cases} p(\pi), & \pi \in \mathbb{P}_T \\ min_{i=1}^n \ v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{P} \\ max_{i=1}^n \ v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{O} \end{cases}$$

Combinatorial games (1): min-max

Let a game arena $S = (\mathbb{P}, \lambda, succ)$ and a payoff function $p : \mathbb{P}_T \to \mathbb{Z} \cup \{+\infty, -\infty\}$ be given.

The *game value* of a position $\pi \in \mathbb{P}$ is traditionally defined as:

$$v_p(\pi) = \begin{cases} p(\pi), & \pi \in \mathbb{P}_T \\ min_{i=1}^{n} \; v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{P} \\ max_{i=1}^{n} \; v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{O} \end{cases}$$

What if we generalize? let's take a generic $\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ instead of $(\mathbb{Z} \cup \{+\infty, -\infty\}, min, max)$

Combinatorial games (2): $\oplus$-$\odot$

Let a game arena $S = (\mathbb{P}, \lambda, succ)$ and a payoff function $p : \mathbb{P}_T \to \mathbb{U}$ be given.

The *game value* of a position $\pi \in \mathbb{P}$ is defined as:

$$v_p(\pi) = \begin{cases} p(\pi), & \pi \in \mathbb{P}_T \\ \bigoplus_{i=1}^{n} v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{P} \\ \bigodot_{i=1}^{n} v_p(\pi_i), & succ(\pi) = \langle \pi_1 ... \pi_n \rangle, \lambda(\pi) = \mathcal{O} \end{cases}$$

Small-step semantics I

$$[\text{Payoff}]\ \frac{\pi \in \mathbb{P}_T \qquad p(\pi) = v}{\pi \to v}$$

$$[\mathcal{P}\text{-expand}]\ \frac{succ(\pi) = \vec{t} \qquad \lambda(\pi) = \mathcal{P}}{\pi \to \sum \vec{t}}\ \#\vec{t} \geq 1$$

$$[\mathcal{O}\text{-expand}]\ \frac{succ(\pi) = \vec{t} \qquad \lambda(\pi) = \mathcal{O}}{\pi \to \prod \vec{t}}\ \#\vec{t} \geq 1$$

$$[\mathcal{P}\text{-reduce}]\ \frac{}{\sum \vec{t}\langle v_1 \ \ v_2\rangle\vec{z} \to \sum \vec{t}\langle v\rangle\vec{z}}\ v_1 \oplus v_2 = v$$

Small-step semantics II

$$[\mathcal{O}\text{-reduce}]\ \frac{}{\prod \vec{t}\langle v_1 \ \ v_2\rangle\vec{z} \to \prod \vec{t}\langle v\rangle\vec{z}}\ v_1 \odot v_2 = v$$

$$[\text{Return}]\ \frac{}{\Lambda\langle v\rangle \to v}$$

$$[\text{Context}]\ \frac{t \to t'}{C[t] \to_c C[t']}\ \text{for all contexts } C$$

The reflexive-transitive closure of $\to_c$ defines rewrites.

Tropical algebras

Definition (Tropical algebra)

$\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is a *tropical algebra* if

- $\oplus$ and $\odot$ are associative
- $\odot$ distributes over $\oplus$: for all $a, b, c \in \mathbb{U}$
  $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and
  $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$

Slightly simplified: see the paper.

Tropical algebras

Definition (Tropical algebra)

$\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is a *tropical algebra* if

- $\oplus$ and $\odot$ are associative
- $\odot$ distributes over $\oplus$: for all $a, b, c \in \mathbb{U}$
  $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and
  $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$

Slightly simplified: see the paper.

The *min,+* algebra $(\mathbb{Z} \cup \{+\infty\}, min, +)$ is an important example of tropical algebra.

The *Rationality hypothesis*

For any α, β it's easy to see that:

$$\alpha \leq \beta + \alpha$$

The *Rationality hypothesis*

For any α, β it's easy to see that:

$$\alpha \leq \beta + \alpha$$

Or, using only the operations in $(\mathbb{Z} \cup \{+\infty\}, min, +)$:

$$\alpha = min\{\alpha, \beta + \alpha\}$$

The *Rationality hypothesis*

For any α, β it's easy to see that:

$$\alpha \leq \beta + \alpha$$

Or, using only the operations in $(\mathbb{Z} \cup \{+\infty\}, \mathit{min}, +)$:

$$\alpha = \mathit{min}\{\alpha, \beta + \alpha\}$$

Generalizing to $(\mathbb{U}, \oplus, \odot)$ we obtain:

Definition (Rationality)

$\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is *rational* iff for any $\alpha, \beta \in \mathbb{U}$ we have that $\alpha = \alpha \oplus (\beta \odot \alpha)$

Slightly simplified: see the paper.

The *Rationality hypothesis*

For any α, β it's easy to see that:

$$\alpha \leq \beta + \alpha$$

Or, using only the operations in $(\mathbb{Z} \cup \{+\infty\}, min, +)$:

$$\alpha = min\{\alpha, \beta + \alpha\}$$

Generalizing to $(\mathbb{U}, \oplus, \odot)$ we obtain:

Definition (Rationality)

$\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is *rational* iff for any $\alpha, \beta \in \mathbb{U}$ we have that $\alpha = \alpha \oplus (\beta \odot \alpha)$

Slightly simplified: see the paper.

"The player $\mathcal{P}$ prefers α over α worsened by β"

The *Rationality hypothesis*

For any α, β it's easy to see that:

$$\alpha \leq \beta + \alpha$$

Or, using only the operations in $(\mathbb{Z} \cup \{+\infty\}, min, +)$:

$$\alpha = min\{\alpha, \beta + \alpha\}$$

Generalizing to $(\mathbb{U}, \oplus, \odot)$ we obtain:

Definition (Rationality)

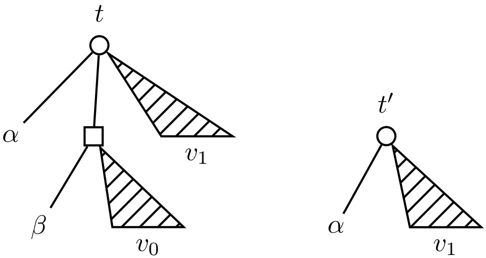
$\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is *rational* iff for any $\alpha, \beta \in \mathbb{U}$ we have that $\alpha = \alpha \oplus (\beta \odot \alpha)$

Slightly simplified: see the paper.

"The player $\mathcal{P}$ prefers α over α worsened by β"
α-pruning depends on **tropicality** and **rationality**

$\mathcal{P}$-cut

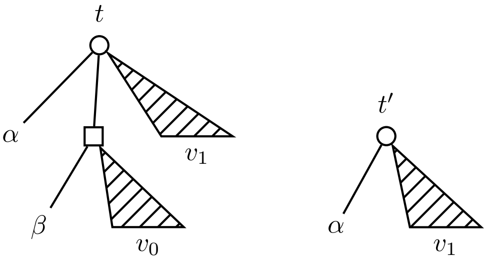
Intuition: think of $\oplus$ as *min* and $\odot$ as $+$. For any v_0, v_1, $\vec{t_0}, \vec{t_1}, \alpha, \beta$ if $\alpha \oplus \beta = \alpha$ then $t \leq t'$ ("t' simulates t"):



"The player $\mathcal{P}$ prefers α to β"

$\mathcal{P}$-cut

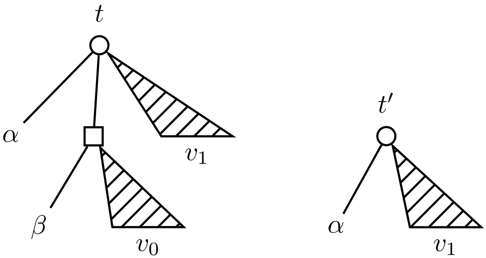
Intuition: think of $\oplus$ as *min* and $\odot$ as $+$. For any v_0, v_1, $\vec{t_0}, \vec{t_1}, \alpha, \beta$ if $\alpha \oplus \beta = \alpha$ then $t \leq t'$ ("t' simulates t"):



"The player $\mathcal{P}$ prefers α to β"
so computing v_0 is a waste of time

$\mathcal{P}$-cut

Intuition: think of $\oplus$ as *min* and $\odot$ as $+$. For any v_0, v_1, $\vec{t_0}, \vec{t_1}, \alpha, \beta$
if $\alpha \oplus \beta = \alpha$ then $t \leq t'$ ("t' simulates t"):

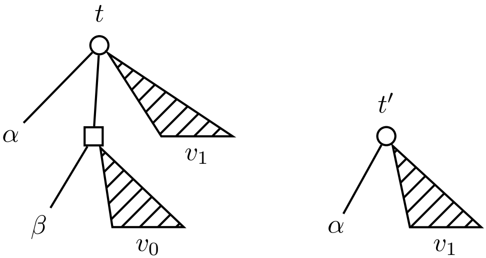


"The player $\mathcal{P}$ prefers α to β"
so computing v_0 is a waste of time

if $min\{\alpha, \beta\} = \alpha$ then $min\{\alpha, (\beta + v_0), v_1\} = min\{\alpha, v_1\}$

$\mathcal{P}$-cut

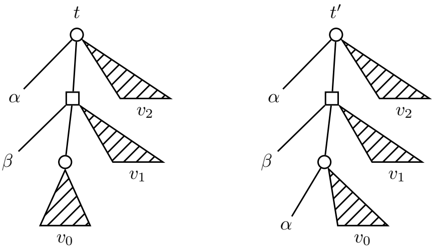
Intuition: think of $\oplus$ as *min* and $\odot$ as $+$. For any v_0, v_1, $\vec{t_0}, \vec{t_1}, \alpha, \beta$
if $\alpha \oplus \beta = \alpha$ then $t \leq t'$ ("t' simulates t"):



"The player $\mathcal{P}$ prefers α to β"
so computing v_0 is a waste of time

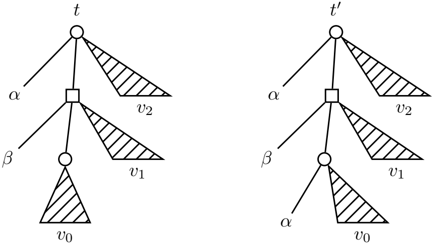
if $min\{\alpha, \beta\} = \alpha$ then $min\{\alpha, (\beta + v_0), v_1\} = min\{\alpha, v_1\}$
if $\alpha \oplus \beta \ = \alpha$ then $\alpha \oplus (\beta \odot v_0) \oplus v_1 = \alpha \oplus v_1$

$\mathcal{P}$-will: for any $\vec{t_0}, \vec{t_1}, \vec{t_2}, \alpha, \beta$ we have $t \leq t'$



"The player $\mathcal{P}$ can bequeath α to her grandchild"
(which allows to prune more)

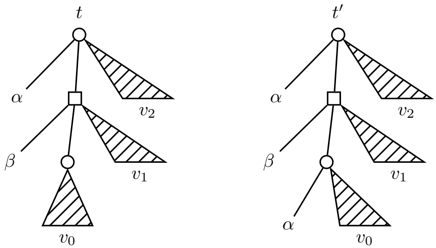
$\mathcal{P}$-will: for any $\vec{t_0}, \vec{t_1}, \vec{t_2}, \alpha, \beta$ we have $t \leq t'$



"The player $\mathcal{P}$ can bequeath α to her grandchild"
(which allows to prune more)

$$min\{\alpha, (\beta + v_0 + v_1), v_2\} = min\{\alpha, (\beta + min\{\alpha, v_0\} + v_1), v_2\}$$

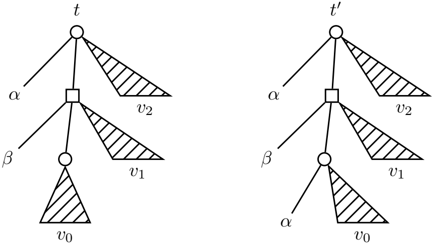
$\mathcal{P}$-will: for any $\vec{t_0}, \vec{t_1}, \vec{t_2}, \alpha, \beta$ we have $t \leq t'$



"The player $\mathcal{P}$ can bequeath α to her grandchild"
(which allows to prune more)

$$min\{\alpha, (\beta + v_0 + v_1), v_2\} = min\{\alpha, (\beta + min\{\alpha, v_0\} + v_1), v_2\}$$
$$\alpha \oplus (\beta \odot v_0 \odot v_1) \oplus v_2 = \alpha \oplus (\beta \odot (\alpha \oplus v_0) \odot v_1) \oplus v_2$$

$\mathcal{P}$-will: for any $\vec{t_0}, \vec{t_1}, \vec{t_2}, \alpha, \beta$ we have $t \leq t'$



"The player $\mathcal{P}$ can bequeath α to her grandchild"
(which allows to prune more)

$$min\{\alpha, (\beta + v_0 + v_1), v_2\} = min\{\alpha, (\beta + min\{\alpha, v_0\} + v_1), v_2\}$$
$$\alpha \oplus (\beta \odot v_0 \odot v_1) \oplus v_2 = \alpha \oplus (\beta \odot (\alpha \oplus v_0) \odot v_1) \oplus v_2$$

{This is due to rationality and to properties of tropical algebras}

Pruning rules (for bi-tropical games)

$$[\mathcal{P}\text{-will}]\ \frac{}{\sum\langle\alpha\ [\prod\langle\beta\ (\sum\vec{t}_1)\rangle\ \vec{t}_2]\rangle\ \vec{t}_3\ \rightarrow\ \sum\langle\alpha\ [\prod\langle\beta\ (\sum\langle\alpha\rangle\vec{t}_1)\rangle\ \vec{t}_2]\rangle\ \vec{t}_3}$$

$$[\mathcal{O}\text{-will}]\ \frac{}{\prod\langle\beta\ [\sum\langle\alpha\ (\prod\vec{t}_1)\rangle\ \vec{t}_2]\rangle\ \vec{t}_3\ \rightarrow\ \prod\langle\beta\ [\sum\langle\alpha\ (\prod\langle\beta\rangle\vec{t}_1)\rangle\ \vec{t}_2]\rangle\ \vec{t}_3}$$

$$[\mathcal{P}\text{-cut}]\ \frac{\alpha\oplus\beta=\alpha}{\sum\langle\alpha\ (\prod\langle\beta\rangle\vec{t}_1)\rangle\ \vec{t}_2\rightarrow\sum\langle\alpha\rangle\vec{t}_2}$$

$$[\mathcal{O}\text{-cut}]\ \frac{\beta\odot\alpha=\beta}{\prod\langle\beta\ (\sum\langle\alpha\rangle\vec{t}_1)\rangle\ \vec{t}_2\rightarrow\prod\langle\beta\rangle\vec{t}_2}$$

α-β-pruning (1) [α best till now for $\mathcal{P}$/min; β best till now for $\mathcal{O}$/max]

```
function alpha_beta(π : ℙ; α, β : ℤ) : ℤ
  if π ∈ ℙ_T then
    return p(π)
  π_1...π_n := succ(π)  # n ≥ 1
  if λ(π) = 𝒫 then
    v := α  # Best for 𝒫 till now
    for i from 1 to n
        and while β <_ℤ v do
      v := min{v, alpha_beta(π_i, v, β)}
  else # λ(π) = 𝒪
    v := β
    for i from 1 to n
        and while v <_ℤ α do
      v := max{v, alpha_beta(π_i, α, v)}
  return v
```

α-β-pruning (2) [α best till now for $\mathcal{P}$/min; β best till now for $\mathcal{O}$/max]

function alpha_beta$(\pi : \mathbb{P};\ \alpha, \beta : \mathbb{Z}) : \mathbb{Z}$
  if $\pi \in \mathbb{P}_T$ **then**
    return $p(\pi)$
  $\pi_1 ... \pi_n := succ(\pi)$ # $n \geq 1$
  if $\lambda(\pi) = \mathcal{P}$ **then**
    $v := \alpha$ # Best for $\mathcal{P}$ till now
    for i **from** 1 **to** n
        and while $\beta <_{\mathbb{Z}} v$ **do** # There's no β
      $v := min\{v, \text{alpha\_beta}(\pi_i, v, \beta)\}$ # There's no β
  else # $\lambda(\pi) = \mathcal{O}$
    $v := \beta$ # There's no β
    for i **from** 1 **to** n
        and while $v <_{\mathbb{Z}} \alpha$ **do**
      $v := max\{v, \text{alpha\_beta}(\pi_i, \alpha, v)\}$
  return v

Tropical α-pruning [α current best for $\mathcal{P}/\oplus$]

function tropical$(\pi : \mathbb{P};\ \alpha : \mathbb{U}) : \mathbb{U}$
  if $\pi \in \mathbb{P}_T$ **then**
    return $p(\pi)$
  $\pi_1 ... \pi_n := succ(\pi)$ # $n \geq 1$
  if $\lambda(\pi) = \mathcal{P}$ **then**
    $v := \alpha$ # Best for $\mathcal{P}$ till now
    for i **from** 1 **to** n **do**
      # do not prune at $\mathcal{P}$'s level
      $v := v \oplus \text{tropical}(\pi_i, v)$
  else # $\lambda(\pi) = \mathcal{O}$
    $v := \text{tropical}(\pi_1, \alpha)$ # No $\mathbf{1}_{\mathbb{U}}$
    for i **from** 2 **to** n
        and while $\alpha \oplus v \neq \alpha$ **do**
      $v := v \odot \text{tropical}(\pi_i, \alpha)$
  return v

Divide and Conquer

Traditional D&C algorithms consist in:

- Divide a problem into sub-problems
- Combine solutions

For example, *merge-sort*.

Divide and Conquer

Traditional D&C algorithms consist in:

- Divide a problem into sub-problems [deterministically]
- Combine solutions

For example, *merge-sort*.

Let's generalize this to *nondeterministic* divisions

Choose-How-To-Divide and Conquer

- non-deterministic choices in a solution space
- some way of "combining" sub-solutions.

Choose-How-To-Divide and Conquer

- non-deterministic choices in a solution space
  choose a division $(\mathcal{P}, \oplus)$: for example *min*
- some way of "combining" sub-solutions.

Choose-How-To-Divide and Conquer

- non-deterministic choices in a solution space
  choose a division $(\mathcal{P}, \oplus)$: for example *min*
- some way of "combining" sub-solutions.
  "accumulating values" $(\mathcal{O}, \odot)$: for example $+$

Choose-How-To-Divide and Conquer

- non-deterministic choices in a solution space
  choose a division $(\mathcal{P}, \oplus)$: for example *min*
- some way of "combining" sub-solutions.
  "accumulating values" $(\mathcal{O}, \odot)$: for example $+$

If $\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is tropical we can use α-pruning

Choose-How-To-Divide and Conquer

- non-deterministic choices in a solution space
  choose a division $(\mathcal{P}, \oplus)$: for example *min*
- some way of "combining" sub-solutions.
  "accumulating values" $(\mathcal{O}, \odot)$: for example $+$

If $\mathcal{A} = (\mathbb{U}, \oplus, \odot)$ is tropical we can use α-pruning

$\oplus$ represents choice according to a quality criterion
$\odot$ "accumulates"

Parsing as a tropical game

Let a context-free grammar be given.

Find the “least-wrong” parse, where each parse may contain unmatched terminals

Parsing as a tropical game

Let a context-free grammar be given.

Find the “least-wrong” parse, where each parse may contain unmatched terminals

- Minimum number of errors
- Minimum number of unmatched characters

Parsing as a tropical game

Let a context-free grammar be given.

Find the "least-wrong" parse, where each parse may contain unmatched terminals

- Minimum number of errors
- Minimum number of unmatched characters

Hypotheses to make exposition simpler (liftable):

- No ϵ-productions
- No two adjacent non-terminals
- No non-terminal alone on a right side

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

- $\mathcal{P}$: Choose a production and a division (pivot characters)

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

- $\mathcal{P}$: Choose a production and a division (pivot characters)
- $\mathcal{O}$: Parse nonterminals “in between” pivots

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

- $\mathcal{P}$: Choose a production and a division (pivot characters)
- $\mathcal{O}$: Parse nonterminals "in between" pivots

$\pi_{\mathcal{P}} = ("1 + 2 + 3", E)$

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

- $\mathcal{P}$: Choose a production and a division (pivot characters)
- $\mathcal{O}$: Parse nonterminals "in between" pivots

$$\pi_{\mathcal{P}} = (\text{"}1+2+3\text{"}, E)$$
$$\pi_{\mathcal{O}} = (\text{"}1\text{"}, E), (\text{"}2+3\text{"}, E)$$

Parsing as a tropical game: nondeterministic parsing

We have a substring to parse with a nonterminal, for example

"1 + 2 + 3" with E.

- $\mathcal{P}$: Choose a production and a division (pivot characters)
- $\mathcal{O}$: Parse nonterminals "in between" pivots

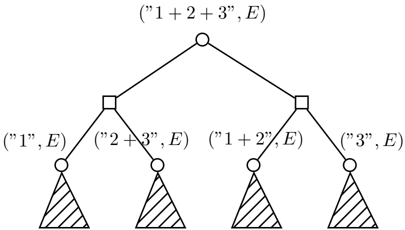
$$\pi_{\mathcal{P}} = (''1 + 2 + 3'', E)$$
$$\pi_{\mathcal{O}} = (''1'', E), (''2 + 3'', E)$$

$$\mathcal{A} = (\mathbb{Z} \cup \{+\infty\}, min, +)$$

Parsing as a tropical game: example

Parsing "1 + 2 + 3":

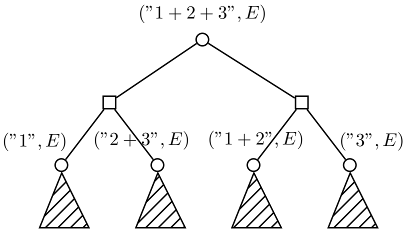
E ::= **n**
E ::= **v**
E ::= **(** E **)**
E ::= **let v =** E **in** E
E ::= **if** E **then** E **else** E
E ::= E **=** E
E ::= E $+$ E
E ::= E $*$ E



Parsing as a tropical game: example

Parsing "1 + 2 + 3":

$E ::= \mathbf{n}$
$E ::= \mathbf{v}$
$E ::= \mathbf{(}\ E\ \mathbf{)}$
$E ::= \mathbf{let}\ \mathbf{v}\ \mathbf{=}\ E\ \mathbf{in}\ E$
$E ::= \mathbf{if}\ E\ \mathbf{then}\ E\ \mathbf{else}\ E$
$E ::= E\ \mathbf{=}\ E$
$E ::= E\ \mathbf{+}\ E$
$E ::= E\ \mathbf{*}\ E$



What if we had **let** instead of **3**? An error...

Memoization

Memoization: avoiding to re-examined repeated positions

Preliminary practical test

Is α-pruning useful?

Is memoization useful?

[little demo]

Other α-pruning examples

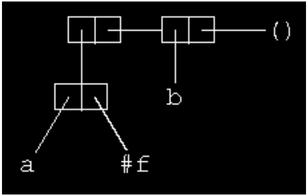
- Searching the most general solution in logic programming [Loddo-DiCosmo, LPAR 2000] [Loddo PhD 2002]

Other α-pruning examples

- Searching the most general solution in logic programming [Loddo-DiCosmo, LPAR 2000] [Loddo PhD 2002]
  - optimization criterion: "less-instantiated terms". *inf* , *sup* bi-tropical

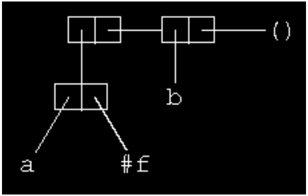
Other α-pruning examples

- Searching the most general solution in logic programming [Loddo-DiCosmo, LPAR 2000] [Loddo PhD 2002]
  - optimization criterion: "less-instantiated terms". *inf* , *sup* bi-tropical
- Draw a planar graph...



Other α-pruning examples

- Searching the most general solution in logic programming [Loddo-DiCosmo, LPAR 2000] [Loddo PhD 2002]
  - optimization criterion: "less-instantiated terms". *inf* , *sup* bi-tropical
- Draw a planar graph...

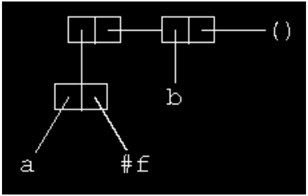


...minimizing:

- the occupied area

Other α-pruning examples

- Searching the most general solution in logic programming [Loddo-DiCosmo, LPAR 2000] [Loddo PhD 2002]
  - optimization criterion: "less-instantiated terms". *inf* , *sup* bi-tropical
- Draw a planar graph...



...minimizing:

- the occupied area
- the number of crossings

Conclusions

We...

- ...defined and proved correct *tropical* α*-pruning*
- ...formalized *min-max* games as particular (bi-)tropical games
- ...modeled *approximated parsing* as a tropical game
  - α-pruning seems to be effective

We would like...

- ...to find more applications
- ...more realistic implementations

Conclusions

We...

- ...defined and proved correct *tropical* α*-pruning*
- ...formalized *min-max* games as particular (bi-)tropical games
- ...modeled *approximated parsing* as a tropical game
  - α-pruning seems to be effective

We would like...

- ...to find more applications
- ...more realistic implementations

Thanks!